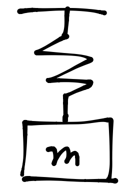
Application of nonhomogeneous ODE : Forced Vibration.

A mass is attached with a spring vertically and is acted by an oscillating external force.



Let u be displacement of the mass from the equilibrium

Equation of motion :

$$mu'' + \gamma u' + ku = F_0 \cos \omega t.$$

$$\left(\text{or } F_1 \cos \omega t + F_2 \sin \omega t\right)$$

I. Undamped case. $\gamma = 0$

$$mu'' + ku = F_0 \cos \omega t.$$

① $\omega \neq \sqrt{\frac{k}{m}}$, complementary sol'n: $u_c = C_1 \cos \sqrt{\frac{k}{m}}\, t + C_2 \sin \sqrt{\frac{k}{m}}\, t$

$\left(\omega_0 = \sqrt{\frac{k}{m}} \text{ natural frequency}\right) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

To find a particular sol'n, set $Y = A \cos \omega t + B \sin \omega t$.

Putting it into the ODE you'll find A, B.

Example: $m = 1$, $k = \omega_0^2$. $u(0) = 0$, $u'(0) = 0$ $F_0 = 1$

$$u'' + \omega_0^2 u = \cos \omega t, \quad u(0) = 0, \; u'(0) = 0.$$

We know the sol'n is

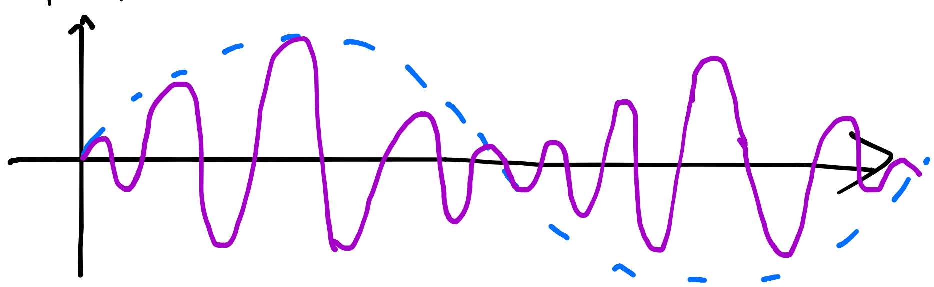
$$u = \frac{1}{\omega_0^2 - \omega^2}\left(\cos \omega t - \cos \omega_0 t\right).$$

When ω is very close to ω_0, we'll observe a phenomenon

Recall: $\cos\alpha - \cos\beta = 2\sin\frac{\beta-\alpha}{2}\sin\frac{\alpha+\beta}{2}$

$$u = \frac{1}{\omega_0^2-\omega^2}\cdot 2\cdot \underbrace{\sin\frac{\omega_0-\omega}{2}t}_{\text{very small.}}\ \underbrace{\sin\frac{\omega_0+\omega}{2}t}_{\text{Roughly } \omega_0.}$$

Graph of $u(t)$.



Such vibration is referred as "beats". Can be used to tune the acoustics.

② $\omega = \sqrt{\frac{k}{m}} = \omega_0$.

$$mu'' + ku = F_0\cos\omega_0 t$$

Comp. sol'n: $u = C_1\cos\omega_0 t + C_2\sin\omega_0 t$

To find a particular sol'n, Set $Y = (A\cos\omega_0 t + B\sin\omega_0 t)t$

b/c exp. coeff. on RHS $= \omega_0 i$. char. roots $= \omega_0 i, -\omega_0 i$

first try fails.

Example: $m=1$, $k=\omega_0^2$, $F_0 = 1$, $u(0)=1$, $u'(0)=0$

$$u'' + \omega_0^2 u = \cos\omega_0 t\ ,\ u(0)=1,\ u'(0)=0.$$

Gen. sol'n: $u = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t + \frac{1}{2\omega_0} t \sin\omega_0 t.$

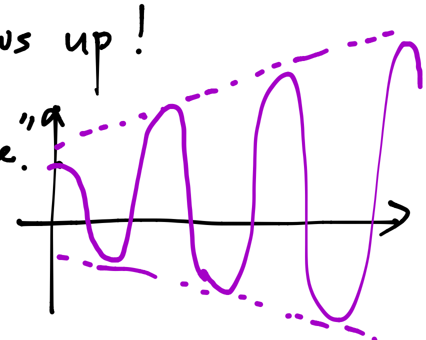
$u(0)=1 \Rightarrow C_1 = 1$

$u'(0)=0 \Rightarrow \omega_0 C_2 = 0 \Rightarrow C_2 = 0.$

$$u(t) = 1\cos\omega_0 t + \frac{1}{2\omega_0} t\sin\omega_0 t = A\cos(\omega_0 t - \varphi)$$

Note that amplitude $A = \sqrt{1 + \frac{t^2}{4\omega_0^2}}$ blows up!

This phenomenon is referred as "resonance".



Case II: Damped case:

$$mu'' + \gamma u' + ku = F_0 \cos\omega t$$

Exp. coeff. on RHS $= \omega i$.

char. root $= -\frac{\gamma}{2m} \pm \frac{\sqrt{|\gamma^2 - 4km|}}{2m} i$. with real part nonzero.

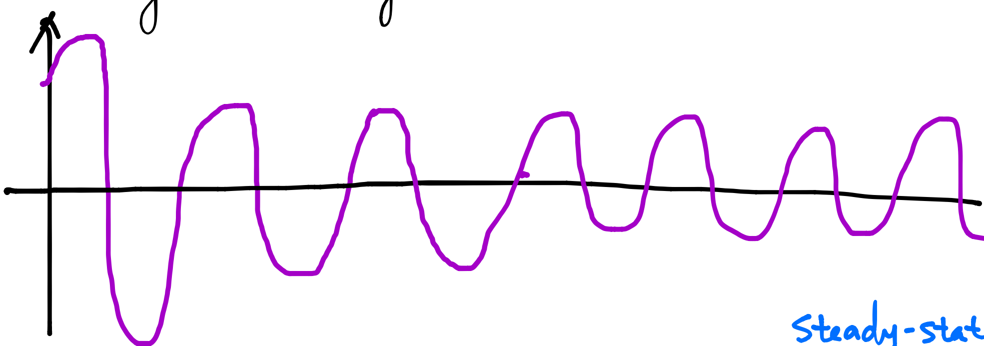
First try succeed. General solution looks like

$$u = C_1 e^{-\frac{\gamma}{2m}t} \cos\frac{\sqrt{|\gamma^2-4km|}}{2m} t + C_2 e^{-\frac{\gamma}{2m}t} \sin\frac{\sqrt{|\gamma^2-4km|}}{2m} t$$ transient sol'n

$$+ A\cos\omega t + B\sin\omega t.$$ Steady-state sol'n.

Note: Amplitude of transient sol'n $\sqrt{C_1^2 + C_2^2}\, \underbrace{e^{-\frac{\gamma}{2m}t}}_{\text{dies out } t\to\infty}$

After a while the vibration contributed by this part will die out. The remaining vibration is contributed only the steady-state sol'n.



Variation of Parameter

For $y'' + p(t)y' + q(t)y = g(t)$ standard form.

the complementary sol'n: $y_c = C_1 y_1 + C_2 y_2$

then a particular solution can be foung using the following formula

$$Y(t) = y_1(t) \int \frac{-y_2(t)\, g(t)}{W(y_1, y_2)} dt + y_2(t) \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)} dt$$

(no need to care about the constants)

How comes the formula:

Idea: Vary the parameters C_1, C_2 in the comp. sol'n.

$$Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t).$$

$$Y'(t) = \underline{u_1'(t)y_1(t)} + u_1(t)y_1'(t) + \underline{u_2'(t)y_2(t)} + u_2(t)y_2'(t)$$

Key simplification: Set $u_1'y_1 + u_2'y_2 = 0$

Reason: We don't want u'' appearing in our computation.

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t)$$

$$Y''(t) = u_1'(t)y_1'(t) + u_1(t)y_1''(t) + u_2'(t)y_2'(t) + u_2(t)y_2''(t)$$

$$Y'' + pY' + qY = g(t)$$

$$= u_1'y_1' + \underline{u_1y_1''} + u_2'y_2' + u_2y_2'' + p(\underline{u_1y_1'} + u_2y_2') + q(\underline{u_1y_1} + u_2y_2)$$

$$= u_1'y_1' + u_2'y_2' + u_1(y_1'' + py_1' + qy_1) + u_2(y_2'' + py_2' + qy_2)$$

$$= u_1'y_1' + u_2'y_2'$$ the rest are zero b/c y_1, y_2 comes from comp. sol'n.

$\Rightarrow$ $u_1'y_1' + u_2'y_2' = g$. or $\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g \end{cases}$

Cramer's rule: $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

Assuming $\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then sol'n to the system

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

(Can be easily proved by direct computation)

Now, our system looks like

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = g \end{cases}$$

$$\Rightarrow u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 g}{W(y_1, y_2)}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 \cdot g}{W(y_1, y_2)}$$

$$\Rightarrow u_1 = \int \frac{-y_2 g}{W(y_1, y_2)} dt, \quad u_2 = \int \frac{y_1 g}{W(y_1, y_2)} dt$$

$$\Rightarrow Y = u_1 y_1 + u_2 y_2 = y_1 \int \frac{-y_2 g}{W(y_1, y_2)} dt + y_2 \cdot \int \frac{y_1 g}{W(y_1, y_2)} dt.$$

Example: $y'' - 2y' + y = te^t$

$y_c = C_1 e^t + C_2 te^t \qquad y_1 = e^t, \quad y_2 = te^t \qquad g = te^t$

$$W(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}.$$

$$Y = e^t \int \frac{-te^t \cdot te^t}{e^{2t}} dt + te^t \int \frac{\cancel{e^t} \cdot t\cancel{e^t}}{\cancel{e^{2t}}} dt$$

$$= e^t\int(-t^2)\,dt + te^t\cdot\int t\,dt$$

$$= e^t\cdot\left(-\frac{1}{3}t^3\right) + te^t\cdot\frac{1}{2}t^2 = \frac{1}{2}t^3e^t - \frac{1}{3}t^3e^t = \frac{1}{6}t^3e^t.$$

Example: $y'' - 2y' + y = \dfrac{e^t}{1+t^2}$

Warning: Don't you dare to use method of undetermined coeffs.

Comp. sol'n: $y_c = C_1e^t + C_2te^t.$

$$y_1 = e^t,\quad y_2 = te^t,\quad g = \frac{e^t}{1+t^2},\quad W(y_1, y_2) = e^{2t}$$

$$u_1 = \int\frac{-te^t\cdot\frac{e^t}{1+t^2}}{e^{2t}}\,dt = \int\frac{-t}{1+t^2}\,dt = -\frac{1}{2}\ln|1+t^2|$$

u-sub
$u = 1+t^2.$

$$u_2 = \int\frac{e^t\cdot\frac{e^t}{1+t^2}}{e^{2t}} = \int\frac{1}{1+t^2}\,dt = \arctan t$$

$$Y = u_1y_1 + u_2y_2 = -\frac{1}{2}\left(\ln|1+t^2|\right)e^t + (\arctan t)\,te^t.$$

Example: $x^2y'' - 3xy' + 4y = x^2\ln x$

Char. eqn: $r(r-1) - 3r + 4 = 0 \Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow r = 2, 2$

Comp. sol'n: $y_c = C_1x^2 + C_2x^2\ln x$

$y_1 = x^2,\quad y_2 = x^2\ln x.\qquad y'' - \dfrac{3}{x}y' + \dfrac{4}{x^2}y = \ln x.$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x^2\cdot\frac{1}{x} \end{vmatrix} = x^3.\qquad g = \ln x.$$

$$u_1 = \int \frac{-y_2 g}{W(y_1, y_2)}\,dx = \int \frac{-x^2 \ln x \cdot \ln x}{x^3}\,dx = -\int \frac{(\ln x)^2}{x}\,dx$$

$$= -\frac{1}{3}(\ln x)^3$$

u-sub
$u = \ln x.$

$$u_2 = \int \frac{y_1 g}{W(y_1, y_2)}\,dx = \int \frac{x^2 \ln x}{x^3}\,dx = \int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2$$

$$Y = u_1 y_1 + u_2 y_2 = -\frac{1}{3}(\ln x)^3 \cdot x^2 + \frac{1}{2}(\ln x)^2 \cdot x^2 \ln x$$

$$= \frac{1}{6} x^2 (\ln x)^3.$$

Gen. sol'n: $y = C_1 x^2 + C_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3.$

Example: Knowing $y_1 = e^t$ is a sol'n to the homog. ODE

$$ty'' - (1+t)y' + y = 0$$

Find the general solution of

$$ty'' - (1+t)y' + y = t^2 e^{2t}.$$

Std. form: $y'' - \frac{1+t}{t} y' + \frac{1}{t} y = t e^{2t}$.

Recall: $y_2 = u(t)\, y_1(t)$, $y_1 u'' + (2y_1' + p y_1) u' = 0.$

$y_2 = u \cdot e^t.$ $\cancel{e^t} u'' + (2\cancel{e^t} - \frac{1+t}{t}\cancel{e^t}) u' = 0$

$$u'' + \left(1 - \frac{1}{t}\right)u' = 0 \Rightarrow \frac{u''}{u'} = \frac{1}{t} - 1$$

$$\Rightarrow \ln|u'| = \ln|t| - t \Rightarrow u' = te^{-t}.$$

$$u = \int \underbrace{t}_{d}\,\underbrace{e^{-t}}_{I}\,dt = -e^{-t}\cdot t + \int (+e^{-t})\cdot dt = -te^{-t} - e^{-t}$$

$$y_2 = u\cdot y_1 = (-te^{-t} - e^{-t})\cdot e^{t} = -t-1.$$

Comp. sol'n: $y_c = C_1 e^{t} + C_2(-t-1) = C_1 e^{t} + C_2(t+1)$

$$y_1 = e^{t}, \quad y_2 = t+1, \quad g = te^{2t}. \quad W(e^t, t+1) = \begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix} = -te^{t}$$

$$u_1 = \int \frac{-(t+1)te^{2t}}{te^{t}}\,dt = -\int \underbrace{(t+1)}_{d}\,\underbrace{e^{t}}_{i}\,dt = -\left((t+1)e^{t} - \int e^{t}\,dt\right) = -te^{t}$$

$$u_2 = \int \frac{e^{t}\cdot te^{2t}}{te^{t}}\,dt = \int e^{2t}\,dt = \frac{1}{2}e^{2t}$$

$$Y = u_1 y_1 + u_2 y_2 = e^{t}\cdot(-te^{t}) + (t+1)\cdot\frac{1}{2}e^{2t} = \frac{1}{2}(1-t)e^{2t}$$

Gen. sol'n: $y = C_1 e^{t} + C_2(t+1) + \frac{1}{2}(1-t)e^{2t}.$

LECTURE NOTES OF DIFFERENTIAL EQUATION

Lecture	Page

Lecture Notes of Differential Equation

Lecture	Page

Lecture Notes of Differential Equation

Lecture	Page